The Surface Domination Number in Plane Graphs

K.A.Kamagnchari [1], Dr.K.Amuthavalli [2], Dr.V.Mohanaselvi [3]
PG and Research Department of Mathematics, Government Arts College,
Ariyalur-621713, Tamil Nadu[1]
Department of Mathematics, Government Arts and Science College,
Vepanthattai,Perambalur-621116, Tamil Nadu[2]
PG and Research Department of Mathematics, Nehru Memorial College(Autonomous),
Puthanampatti,Tiruchirappalli-621007, Tamil Nadu[3]
Email: meetkak85@gmail.com[1], thrcka@gmail.com[2], vmohanaselvi@gmail.com[3]

Abstract-The domination concept in simple graph is extended to the surface of the plane graph and hence introduced the surface domination in plane graphs. Let $\mathrm{G_P} = \{V, E, S\}$ be a plane graph where V, E and S are respectively denote the vertex, edge and surface set of $\mathrm{G_P}$ with order m, size n and face r. Then a surface subset $\mathrm{R}(\subseteq \mathrm{S})$ is said to be a surface dominating set of $\mathrm{G_P}$ if every surface in $\mathrm{S}-\mathrm{R}$ there exist at least one surface in R such that they are adjacent in $\mathrm{G_P}$. Also, the surface dominating set is characterized and the bounds of the parameters are presented in basic theoretical terms.

Index Terms- Plane graph; Surface Domination; Surface Domination Number.

1. INTRODUCTION

Let $G = (V, E)$ be a simple (m, n) graph. A graph is said to be embedded in a surface S when it is drawn on S so that no two edges intersect. In 1758, Euler discovered the planar graph from the investigation of polyhedra and given a result that for any spherical polyhedron with m vertices, n edges and f faces, $m - n + f = 2$ which is known as Euler Polyhedron Formula. A plane map is a connected plane graph together with all its faces. A graph is planar if it can be embedded in a plane. A plane graph $G_P = (V, E, S)$ where V, E and S are respectively denote the vertex, edge and surface set of G_P is a graph drawn in the plane in such a way that no pair of edges intersect. Since plane graph is embedded in the plane gives every plane graph is a planar graph. The regions defined by a plane graph referred as its faces. The unbounded region is called the exterior face and the surface bounded by a cycle is called the interior face of G_P. Hence G_P has exactly one exterior face. A face is said to be surface if it is not an exterior face. All the interior faces are named as surfaces. Two surfaces are said to be adjacent if they have the same edge as their one of the boundary else they are said to be non-adjacent. The face r of G_P is the number of surfaces in G_P.

In 1958, Berge defined the concept of the domination number for a graph calling that as "Coefficient of External Stability". In 1962, Ore used the name "dominating set" and "domination number" for the same concept. In 1977, Cockayne and Hedetniemi extended many results about dominating sets in graphs. For a complete review on the topic domination and its related parameter appears in [6]. A dominating set D of vertices in a graph is called a dominating set of G if every vertex in $V - D$ is adjacent to a vertex in D. The domination number denoted as $\gamma(G)$ is the minimum cardinality of all dominating sets of G.

In [9], Mitchel and Hedetniemi introduced edge domination in graphs. A set F of edges in a graph G is called an edge dominating set of G if every edge in $E - F$ is adjacent to at least one edge in F. The edge domination number denoted as $\gamma'(G)$ is the minimum cardinality of all edge dominating sets of G.

The degree of a vertex $v \in V(G)$ is the number of edges incident with v and is denoted by $d(v)$. The minimum and maximum degree among all the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$. The degree of an edge $e = uv$, where $u, v \in V(G)$ is defined as $d(u) + d(v) - 2$ and is denoted by $d(e)$. The minimum and maximum degree among all the edges of G is denoted by $\delta'(G)$ and $\Delta'(G)$. The degree of a surface $s \in S$ of a plane graph G_P is the number of edges in the boundary of the surface s and is denoted by $d(s)$. The minimum and maximum degree among all the surfaces of G_P is denoted by $\delta_s(G_P)$ and $\Delta_s(G_P)$. A plane graph G_P is said to be connected if there exist a path between any two vertices otherwise disconnected.

In this paper, we introduced the surface domination for plane graphs and hence defined the surface dominating set and the corresponding surface domination number. Further, we characterized the surface dominating sets and found its bounds.

2. PRELIMINARIES

Definition 2.1 Let $G_p = (V, E, S)$ be a plane graph with vertex set $V(G_P) = \{v_1, v_2, \dots, v_m\}$ of order m, edge set $E(G_P) = \{e_1, e_2, \dots, e_n\}$ of size n and surface set $S(G_P) = \{s_1, s_2, \dots, s_r\}$ of face r denoted by (m, n, r) graph. A surface number S_N of a surface s_i in G_P is the number of surfaces adjacent with s_i either by vertices or by edges and is denoted by $S_N(s_i)$.

Definition 2.2. A surface s_m in G_P is a mono surface if exactly one vertex of s_m is common with any other surfaces in G_P. Hence $S_N(s_m) = 1$.

Definition 2.3. A surface s_l in G_P is said to be an island if either of its vertices or edges are not common with any other surfaces in G_P. Hence $S_N(s_l) = 0$.

Definition 2.4. Let G_P be a plane graph, then for any surface $s_i \in S$, the open neighbourhood or neighbourhood of s_i is the set $N(s_i) = \{s_j \in S \,|\, s_i \text{ and } s_j \text{ are adjacent in } G_P\}$ and closed neighbourhood of s_i is a set $N[s_i] = N(s_i) \cup \{s_i\}$.

Example 2.5.

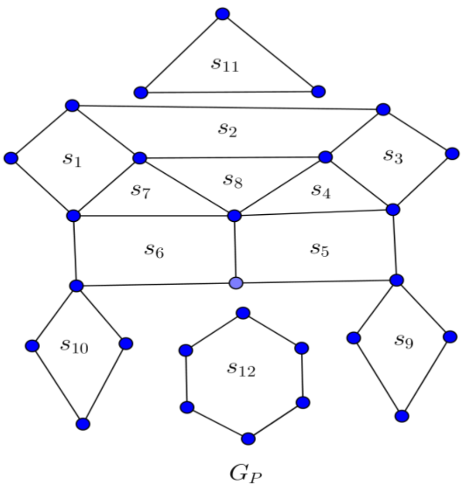

Fig. 2.1

For the above graph G_P in Figure 2.1 we have, the surface number of every surfaces are $S_N(s_1) = S_N(s_3) = 4$, $S_N(s_2) = 5$, $S_N(s_4) = S_N(s_5) = S_N(s_6) = S_N(s_7) = 6$, $S_N(s_8) = 7$, $S_N(s_9) = S_N(s_{10}) = 1$, $S_N(s_{11}) = S_N(s_{12}) = 0$. The mono surfaces of G_P are s_9 and s_{10}.
The islands are s_{11} and s_{12}.
The open and closed neighbourhood set of s_5 is $\{s_4, s_6\}$ and $\{s_4, s_5, s_6\}$.
Also note that, the open and closed neighbourhood set of a mono surface s_9 is $\{\emptyset\}$ and $\{s_9\}$. For the island there is no open and closed neighbourhood set.

Definition 2.6. A plane graph G_p without mono surface is called simple plane graph.

Observation 2.7. In a plane graph G_P**,** a surface which has a surface number one need not be a mono surface.
For example, in the following plane graph G_P in Figure 2.3 we have, $S_N(s_1) = 1 = S_N(s_2)$, but they are not mono surfaces.

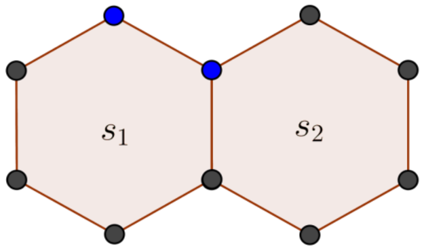

Fig. 2.2

Remark 2.8.

Through out this paper, we consider a plane graph G_P is surface structured graph. It means that G_P consist of only cycles in which the cycles are detached or attached by another cycle either by edges or by vertices. Hence minimum two edges are needed to make a disconnected plane graph with two component to become a connected plane graph.

Theorem 2.9. (Euler's theorem on surfaces): If G_P is a (m, n, r) connected plane graph with at least two surfaces then, $m - n + r = 1$.

Proof: Let the surface set of G_P be $S = \{s_1, s_2, \dots, s_r\}$ such that s_i has m_i number of vertices and n_i number of edges for $1 \le i \le r$. Let k be the number of mono surfaces in G_P. Then the following cases arises in G_P.
case (1): $k = 0$.
In this case the connected plane graph G_P becomes simple plane graph.
Then G_P has $m = \sum_{i=1}^{r} m_i - 2(r-1)$ number of vertices, $n = \sum_{i=1}^{r} n_i - (r-1)$ number of edges and r number of surfaces.

Therefore, $m - n + r = \sum_{i=1}^{r} m_i - 2(r-1) - [\sum_{i=1}^{r} n_i - (r-1)] + r$.
Since all surfaces are cycle gives $\sum_{i=1}^{r} m_i = \sum_{i=1}^{r} n_i$.
Hence $m - n + r = 1$.

case (2): $k = t, 1 \leq t < r$.
In this case G_P has face $r = h + t$, where h is a number of non-mono surfaces.
Then G_P has $m = \sum_{i=1}^{r} m_i - 2(h-1) - t$ number of vertices, $n = \sum_{i=1}^{r} n_i - (h-1)$ number of edges and $h + t$ number of surfaces.
Therefore, $m - n + r = [\sum_{i=1}^{r} m_i - 2(h-1) - t] - [\sum_{i=1}^{r} n_i - (h-1)] + (h+t)$.
Hence $m - n + r = 1$.

case (3): $k = r$.
In this case all the surfaces in G_P are mono surfaces.
Then G_P has $m = \sum_{i=1}^{r} m_i - (r-1)$ number of vertices, $n = \sum_{i=1}^{r} n_i$ number of edges and r number of surfaces.
Therefore, $m - n + r = \sum_{i=1}^{r} m_i - (r-1) - \sum_{i=1}^{r} n_i + r$.
Hence $m - n + r = 1$. □

Corollary 2.10. For any (m, n, r) plane graph G_P with k components, $m - n + r = k$.

Proof: Since every component is a connected plane graph then by the above Theorem 2.9 we have, $m - n + r = 1$. Hence for the component $i, m_i - n_i + r_i = 1$, for $1 \leq i \leq k$. Therefore, for k components we have, $\sum_{i=1}^{k} m_i - \sum_{i=1}^{k} n_i + \sum_{i=1}^{k} r_i = \sum_{i=1}^{k} 1$. It gives $m - n + r = k$. □

3. SURFACE DOMINATION

Definition 3.1. Let $G_P = (V, E, S)$ be a (m, n, r) plane graph. A surface set $R \subseteq S(G_P)$ is said to be a surface dominating set of G_P if for every surface s_i in $S - R$ there exist at least one surface s_j in R such that s_i and s_j are adjacent. The surface domination number is defined as the minimum number of surfaces that dominates the graph G_P and is denoted by $\gamma_{\S}(G_P)$.

Example 3.2.
For graph G_{P_1} in Figure 3.1 we have $S1 = \{s_1, s_3, s_4, s_5, s_7\}$, $S2 = \{s_1, s_4, s_5, s_7\}$, $S3 = \{s_1, s_3, s_6, s_7\}$, $S4 = \{s_2, s_4, s_6\}$, $S5 = \{s_3, s_6\}$ are some surface dominating set. Since $S4$ has minimum cardinality gives $\gamma_{\S}(G_{P_1}) = 2$. Also, note that graph G_{P_2} is a graph with all surfaces are mono surfaces and graph G_{P_3} is disconnected.

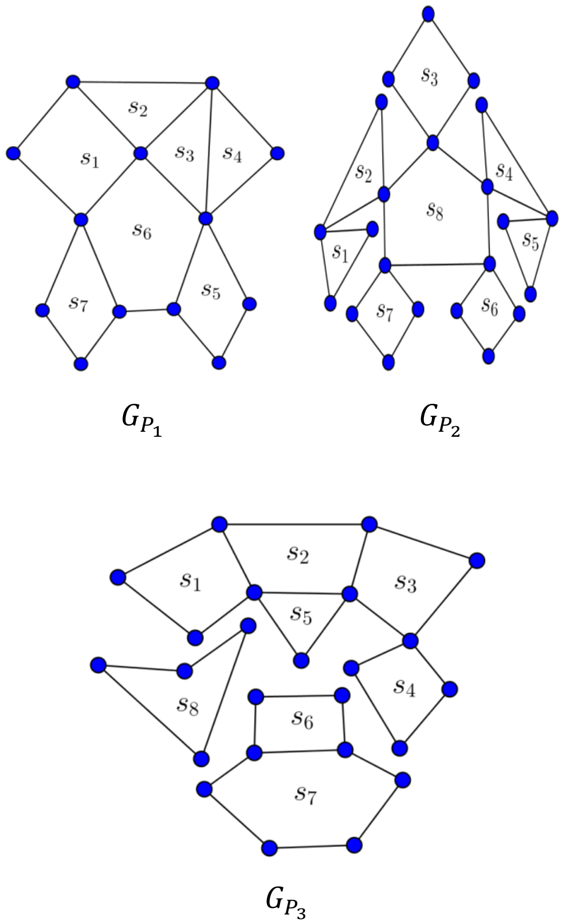

Fig. 3.1

Definition 3.3. A surface dominating set R of G_P is minimal surface dominating set if no proper subset of R is a surface dominating set of G_P.
For graph G_{P_1} in Figure 3.1 the sets $S2$, $S4$ and $S5$ are minimal surface dominating sets.

Remark 3.4. For the study of the surface domination number throughout this paper we consider the plane graph G_P has at least one surface dominating set.

Theorem 3.5. Let G_P be a simple connected plane graph. Then a surface dominating set R of G_p is a minimal dominating set if and only if for each surface $s \in R$ one of the following condition holds:
(i) s is a mono surface or an island.
(ii) There exist an surface $r \in S - R$ such that $N(s) \cap R = \{s\}$ or $\emptyset$.
(iii) There exist a surface r in R such that s and r have a common vertex.

Proof: Assume R is a minimal dominating set of G_p. Then for every surface s in R, $R-\{s\}$ is not a surface dominating set and hence there will be some surface r in $S-R\cup\{s\}$ is not surface dominated by any surface in $R-\{s\}$. This gives either $r=s$ or $r\in S-R$. If $r=s$ then s is a mono surface or an island. If surface $r\in S-R$ then r is not surface dominated by $R-\{s\}$, but is surface dominated by elements of R, then r is adjacent only to the surface s in R gives $N(r)\cap R=\{s\}$ or $\emptyset$. If there is a surface r in R and $r\neq s$ then the surfaces can be disjoint or connected. If it is connected it can be connected by a vertex or edges. If r and s are not adjacent then, they have a common vertex. Conversely, assume the conditions (i), (ii) and (iii) hold. Suppose R is not a minimal dominating set. Then there exist a surface s in R such that $R-\{s\}$ is a surface dominating set. Then (i) does not hold. If $R-\{s\}$ is a surface dominating set then every surface in $S-(R-\{s\})$ is adjacent to at least one surface in $R-\{s\}$ then we have that (ii) and (iii) does not hold. This contradicts our assumption and hence the converse. □

Theorem 3.6. Let G_P be a simple connected plane graph. Then for every minimal surface dominating set R in G_P, $S-R$ is also an surface dominating set of G_P.

Proof: Let R be a minimal surface dominating set of G_P. Suppose $S-R$ is not a surface dominating set. Then there exists a surface s in S such that s is not surface dominated by any surface in $S-R$. Since G_P is a simple connected plane graph, s is surface dominated by at least one surface in $R-\{s\}$ which contradict the surface minimality of R. Hence $S-R$ is a surface dominating set. □

From the definition of minimal surface dominating set and minimum surface dominating set we have the following theorem is obvious.

Theorem 3.7. Every minimum surface dominating set in G_p is a minimal surface dominating set.

The converse of the above theorem is not true. For example, in the plane graph G_{P_1} in Figure 3.1, the surface set $\{s_1,s_4,s_5,s_7\}$ is a minimal surface dominating set, but it is not a minimum surface dominating set.

The following theorem gives the surface domination number in terms of face value:

From the definition of surface domination number the following bounds are obvious.

Theorem 3.8. G_P is a (m,n,r) plane graph with at least one surface, then $1\leq \gamma_{\S}(G_P)\leq r$.

Theorem 3.9. For any (m,n,r) connected plane graph, the surface domination number is r if and only if all its surfaces are mono surface.

Proof: Let G_P be a connected plane graph with r surfaces. Since all the surfaces are mono surfaces and the surfaces are connected each other only by a vertex and not by edges. No two surfaces are adjacent. Then all the surfaces are in the minimum surface dominating set and hence $\gamma_{\S}(G_P)=r$.

On the otherhand, suppose $\gamma_{\S}(G_P)=r$, let R be a minimum surface dominating set with cardinality r. Now, prove that G_P has only mono surfaces. Conversely, assume that G_P has at least one non mono surface. Since G_P is connected gives R has at most $r-1$ surfaces which contradict the minimality of R. Hence all the surfaces in G_P are mono surfaces. □

Theorem 3.10. For any *(m, n, r)* simple connected plane graph G_P, $\gamma_{\S}(G_P)\leq \frac{r}{2}$.

Proof: Let R be a minimal surface dominating set of G_P. By Theorem 3.6, $S-R$ is a surface dominating set of G_P. Hence $\gamma_{\S}(G_P)\leq \min\{|R|,|S-R|\}\leq \frac{r}{2}$, where $|R|$ and $|S-R|$ are cardinality of R and cardinality of $S-R$. □

Theorem 3.11. For any simple connected *(m, n, r)* plane graph G_P, $\gamma_{\S}(G_P)\geq \frac{r}{2\Delta_s(G_P)}$, where $\Delta_s(G_P)$ is a maximum surface degree of G_P.

Proof: Let R be a surface dominating set with cardinality $\gamma_{\S}(G_P)$. Then it is obvious that $|R|\Delta_s\geq \frac{r}{2}$. It gives $\gamma_{\S}(G_P)\geq \frac{r}{2\Delta_s(G_P)}$. □

Theorem 3.12.[5] If G is any planar (m,n) graph with $m\geq 3$, then $n\leq 3m-6$. Furthermore, if G has no triangles, then $n\leq 2m-4$.

Theorem 3.13. For any (m,n,r) connected plane graph G_P, $r\leq 2m-5$, $m\geq 3$ and if G_P has no triangles, then $r\leq m-3$.

Proof: Since G_P is a connected (m, n, r) plane graph. Then, by Theorem 2.9 and Theorem 3.12, we have $r = 1 + n - m$ and $n \leq 3m - 6$ gives $r \leq 2m - 5$. Similarly, if G_P has no triangles, then $n \leq 2m - 4$ gives $r \leq m - 3$. □

Theorem 3.10 and 3.13 gives the following results.

Corollary 3.14. For any (m, n, r) simple connected plane graph G_P, $\gamma_{\S}(G_P) \leq (2m - 5)/2$ and with no triangles, $\gamma_{\S}(G_P) \leq (m - 3)/2$.

4. CONCLUSION:

In this paper the surface domination is introduced and the corresponding surface dominating set and surface domination number are defined for simple graph. Also, the characterizations of the surface dominating set are presented. Also the bounds and exact values of the surface domination number are given. Further, the surface domination parameter can be extended by studying the properties like independent, connected and disconnected, etc.

REFERENCES

[1] Arumugam,.S,; Velammal,.S. (1998): 'Edge domination in graphs', Taiwanese Journal of Mathematics, 2(2), pp.173-179.

[2] Berge, C. (1962): Theory of Graphs and its Applications, Methuen, London.

[3] Cockayne, E.J.; Hedetniemi,.S.T. (1977): 'Towards a theory of domination in graphs', Networks, 7, pp.247-261.

[4] Douglas B.West, Introduction to Graph Theory, 2^{nd} Edition, University of Illinois-Urbane- PHI-Pvt. Ltd., New Delhi.

[5] Harary, F. (2001): Graph Theory, Addison-Wesley Publishing company.

[6] Haynes, T.W.; Hedetniemi, S.T.; Slater, P.J. (1998): Fundamentals of Domination in Graphs. Marcel Dekker, New York.

[7] John Clark Derek Allan Holton, A First Look At Graph Theory, Allied publ., Ltd.

[8] Kulli, V.R. (2010): Theory of Domination in Graphs, Vishwa International Pub., India.

[9] Mitchell, S & Hedetniemi, S.T. (1977): 'Edge domination in trees', Congr. Numer. 19, pp.489-509.

[10] Preeti Gupta, March 2013: 'Domination in Graph with Application', Indian Journal of Research, Paripex, Volume: 21 Issue: 3, ISSN-2250-1991.

(A.1)